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On the Denseness of Rational Systems

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This note characterizes the denseness of rational systems

$$\mathscr{P}_{n-1}(a_1, ..., a_n) := \left\{ \frac{P(x)}{\prod_{k=1}^{n} (x-a_k)}, P \in \mathscr{P}_{n-1} \right\} \qquad (n=1, 2, ...),$$

in $C[-1, 1]$, where the nonreal poles in $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C} \setminus [-1, 1]$ are paired by complex conjugation. This extends an Achiezer's result.

Key Words: denseness; rational system; prescribed poles; Chebyshev polynomials.

1. INTRODUCTION

We let

$$\mathscr{P}_m(a_1; ..., a_n) := \left\{ \frac{P(x)}{\sum_{k=1}^{n} |x-a_k|}, P \in \mathscr{P}_m \right\} \tag{1.1}$$

with $\{a_k\}_{k=1}^{n} \subset \mathbb{C} \setminus [-1, 1]$, where $\mathscr{P}_m$ is the set of all real algebraic polynomials of degree at most m. It is easy to see that $\mathscr{P}_m(a_1, ..., a_n)$ is a linear space and $\mathscr{P}_m(a_1, ..., a_n) \subset \mathscr{P}_M(a_1, ..., a_n)$ for $m < M$. We define the numbers $\{c_k\}_{k=1}^{n}$ by

$$a_k := \frac{c_k + c_k^{-1}}{2}, \qquad |c_k| < 1. \tag{1.2}$$

When all the poles $\{a_k\}_{k=1}^{n}$ are real and distinct, $\mathscr{P}_{n-1}(a_1, a_2, ..., a_n)$ is simply the real span of the following system

$$\left\{ \frac{1}{x-a_1}, \frac{1}{x-a_2}, ..., \frac{1}{x-a_n} \right\}, \qquad x \in [-1, 1]. \tag{1.3}$$

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With respect to the denseness of $\text{span}\{1/(x-a_k)\}_{k=1}^{\infty}$, the following well-known result is due to Achiezer [1, Problem 7, p. 254]:

ACHIEZER THEOREM. *Let* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R}\setminus[-1, 1]$ *be distinct. Then* $\text{span}\{1/(x-a_k)\}_{k=1}^{\infty}$ *is dense in* $C[-1, 1]$ *if and only if*

$$\sum_{k=1}^{\infty} (1-|c_k|) = \infty.$$

Recently, Borwein and Erdélyi [3] also proved this by using entirely different methods.

Note that $\mathscr{P}_{n-1}(a_1, ..., a_n)$ is still a real rational space when the nonreal poles form complex conjugate pairs, moreover, $\prod_{k=1}^{n} |x-a_k|$ can be replaced by $\prod_{k=1}^{n} (x-a_k)$. So, it is natural to ask whether we can extend Achiezer's Theorem to the case: the repeated poles are allowed and the nonreal elements in $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C}\setminus[-1, 1]$ are paired by complex conjugation.

In this note, we consider this question and give an affirmative answer. More precisely, we have

THEOREM 1.1. *Let the nonreal elements in* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C}\setminus[-1, 1]$ *be paired by complex conjugation. Then* $\{\mathscr{P}_{n-1}(a_1, ..., a_n)\}$ *are dense in* $C[-1, 1]$ *if and only if*

$$\sum_{k=1}^{\infty} (1-|c_k|) = \infty. \tag{1.4}$$

2. PROOF OF THEOREM 1.1

Our proof of Theorem 1.1 is mainly based on the Chebyshev polynomials with respect to $\mathscr{P}_n(a_1, ..., a_n)$ constructed recently by Borwein, Erdélyi, and Zhang [4]. The explicit formulae for the Chebyshev polynomials for the system $\mathscr{P}_n(a_1, a_2, ..., a_n)$ are implicitly contained in Achiezer [1] provided that $\{a_k\}_{k=1}^{n} \subset \mathbb{R}\setminus[-1, 1]$ are distinct. It should be mentioned that they [4] allow *repeated poles* and *nonreal poles* in this system, in which the nonreal poles form complex conjugate pairs (cf. [4]). We use $T_n(x)$ to denote the Chebyshev polynomial of the first kind with respect to $\mathscr{P}_n(a_1, a_2, ..., a_n)$. For convenience, we include its construction here.

Let

$$M_n(z) := \left(\prod_{k=1}^{n} (z-c_k)(z-\bar{c}_k)\right)^{1/2}, \tag{2.1}$$

where the square root is defined so that $M_n^*(z) = z^n M_n(z^{-1})$ is analytic in a neighbourhood of the closed unit disk, and let

$$f_n(z) := \frac{M_n(z)}{z^n M_n(z^{-1})}. \tag{2.2}$$

Then the *Chebyshev polynomial of the first kind* for the rational space $\mathcal{P}_n(a_1, a_2, ..., a_n)$ is defined by

$$T_n(x) := \frac{f_n(z) + 1/f_n(z)}{2}, \qquad x = \frac{z + z^{-1}}{2}, \qquad |z| = 1. \tag{2.3}$$

In fact $T_n(x)$ is a rational function. More precisely, we conclude that $T_n \in \mathcal{P}_n(a_1, ..., a_n)$ (cf. [4, Theorem 1.2]). It is shown [4] that these Chebyshev polynomials preserve almost all the elementary properties of the classical Chebyshev polynomials.

LEMMA 2.1. *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1, 1]$ *be paired by complex conjugation, and let* T_n *be the Chebyshev polynomial of the first kind associated with* $\mathcal{P}_n(a_1, ..., a_n)$. *Then the best approximation to* 1 *from* $\mathcal{P}_{n-1}(a_1, a_2, ..., a_n)$ *is*

$$p := 1 - T_n / A_0. \tag{2.4}$$

Moreover, we have

$$\|1 - p\|_{[-1, 1]} = 1/|A_0|, \tag{2.5}$$

where A_0 *is the constant term in* T_n:

$$A_0 := \frac{(-1)^n}{2} \Big((c_1 \cdots c_n)^{-1} + c_1 \cdots c_n \Big). \tag{2.6}$$

Proof. Clearly, there exists some $r \in \mathcal{P}_{n-1}(a_1, ..., a_n)$ such that

$$T_n(x) := A_0 + r(x).$$

Then we conclude that

$$A_0 = \lim_{x \to \infty} T_n(x),$$

furthermore, by the construction of T_n, it is easy to show (2.6) (cf. [4, Proposition 4.1]). The conclusions of (2.4) and (2.5) can be proved by the same fashion as Lemma 2.2, that is by the counting zeros' argument. We omit it. ∎

Let $a \in \mathbb{R} \backslash [-1, 1]$ such that $a \notin \{a_k\}_{k=1}^{n}$, then we define the constant c such that

$$a = \frac{c + c^{-1}}{2}, \qquad |c| < 1. \tag{2.7}$$

Let T_{n+1} be the Chebyshev polynomial of the first kind with respect to $\mathscr{P}_{n+1}(a_1, ..., a_n, a)$. Lemma 2.2 gives the best approximation to $1/x - a$ from $\mathscr{P}_n(a_1, ..., a_n)$.

LEMMA 2.3. *Let the nonreal elements in* $\{a_k\}_{k=1}^{n} \subset \mathbb{C} \backslash [-1, 1]$ *be paired by complex conjugation. Then, for* $a \in \mathbb{R} \backslash [-1, 1]$ *and* $a \notin \{a_k\}_{k=1}^{n}$, *the best approximation to* $1/x - a$ *from* $\mathscr{P}_n(a_1, ..., a_n)$ *on* $[-1, 1]$ *is*

$$q := \frac{1}{x - a} - \frac{T_{n+1}(x)}{B_{n+1}} \tag{2.8}$$

and

$$\left\| \frac{1}{x - a} - q(x) \right\|_{[-1, 1]} = \frac{1}{|B_{n+1}|}, \tag{2.9}$$

where

$$B_{n+1} := -\left(\frac{c - c^{-1}}{2}\right)^2 \prod_{j=1}^{n} \frac{1 - cc_j}{c - c_j}. \tag{2.10}$$

Proof. We prove it by the counting zeros' argument. Since $a \in \mathbb{R} \backslash [-1, 1]$ and $a \notin \{a_k\}_{k=1}^{n}$, we then can construct the Chebyshev polynomial of the first kind T_{n+1} for $\mathscr{P}_{n+1}(a_1, ..., a_n, a)$ and it can be expressed as

$$T_{n+1}(x) := s(x) + \frac{B_{n+1}}{x - a},$$

where $r \in \mathscr{P}_n(a_1, ..., a_n)$. Since

$$B_{n+1} = \lim_{x \to a} (x - a)\, T_{n+1}(x),$$

then it is easy to show (2.10) by a simple calculation. Moreover, $q(x) = -s(x)/B_{n+1}$. Note that (cf. [4, Theorem 1.2]) $\|T_{n+1}\|_{[-1, 1]} = 1$, we have

$$\left\| \frac{1}{x - a} - q(x) \right\|_{[-1, 1]} = \frac{1}{|B_{n+1}|}. \tag{2.11}$$

If there exists some $t \in \mathscr{P}_n(a_1, ..., a_n)$ such that

$$\left\| \frac{1}{x-a} - t(x) \right\|_{[-1,1]} < \frac{1}{|B_{n+1}|}, \tag{2.12}$$

recall that (cf. [4, Theorem 1.2]) there exist $n+2$ nodes: $-1 = y_{n+1} < y_n < \cdots < y_1 < y_0 = 1$ such that $T_{n+1}(y_j) = (-1)^j$, $j = 0, ..., n, n+1$. Hence,

$$\frac{T_{n+1}}{B_{n+1}} - \left(\frac{1}{x-a} - t(x) \right) = -q + t \in \mathscr{P}_n(a_1, ..., a_n)$$

changes sign between any two consecutive extreme points of T_{n+1}. Furthermore, $t - q$ has at least $n+1$ zeros in $(-1, 1)$ and consequently, it must vanish identically. This contradicts (2.12). ∎

Proof of Theorem 1.1. We first prove *only if* part. Note that $|c_k| < 1$ $(k = 1, 2, ...)$ and by (2.6) we then have

$$\prod_{k=1}^{n} |c_k| < \frac{1}{|A_0|} = \frac{2 \prod_{k=1}^{n} |c_k|}{1 + \prod_{k=1}^{n} |c_k|^2} \leqslant 2 \prod_{k=1}^{n} |c_k|.$$

If $\{\mathscr{P}_{n-1}(a_1, ..., a_n)\}$ are dense in $C[-1, 1]$, then by Lemma 2.1 we have $1/|A_0| \to 0 (n \to \infty)$, that is, $\prod_{k=1}^{\infty} |c_k| = 0$, this is equivalent to (1.4).

Next we prove *if* part. By (2.10) we have

$$\frac{1}{|B_{n+1}|} \to \left(\frac{2}{c - c^{-1}} \right)^2 \prod_{j=1}^{\infty} \left| \frac{c - c_j}{1 - c c_j} \right| \qquad (n \to \infty).$$

Recall that $\prod_{k=1}^{\infty} (c - c_j)/(1 - c c_j)$ is an infinite Blaschke product. Then by [6, Theorem 1, p. 281] or [5, Theorem 15.23, p. 311] we conclude that (1.4) implies

$$\prod_{j=1}^{\infty} \left| \frac{c - c_j}{1 - c c_j} \right| = 0,$$

consequently, combining (2.9) we see that $1/(x-a)$ can be uniformly approximated in $\{\mathscr{P}_n(a_1, ..., a_n)\}$ on $[-1, 1]$ for $a \in \mathbb{R} \backslash [-1, 1]$. Also, if (1.4) holds, then from the proof of *only if* part and Lemma 2.1, we see that any constant can be uniformly approximated in $\{\mathscr{P}_{n-1}(a_1, ..., a_n)\}$. Note that every function $R \in \mathscr{P}_n(a_1, ..., a_n)$ can be written in the form

$$R(x) = b_n + R_0(x), \qquad b_n \in \mathbb{R}, \qquad R_0 \in \mathscr{P}_{n-1}(a_1, ..., a_n),$$

and $\mathscr{P}_{n-1}(a_1, ..., a_n) \subset \mathscr{P}_{N-1}(a_1, ..., a_N)$ for $n < N$. Thus, (1.4) implies that $1/(x-a)$ can be uniformly approximated in $\{\mathscr{P}_{n-1}(a_1, ..., a_n)\}$ on $[-1, 1]$. Note that $a \in \mathbb{R} \backslash [-1, 1]$ is an arbitrary number, so we can take a to be any of a sequence of distinct number such that they satisfy the condition (1.4), that mean $1/(x-a)$ can be taken as any of a dense sequence of distinct basis functions. Therefore, *if* part follows. ∎

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